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Quantum mechanical fluctuations at the end of inflation

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Abstract

During the inflationary phase of the early universe, quantum fluctuations in the vacuum generate particles as they stretch beyond the Hubble length. These fluctuations are thought to result in the density fluctuations and gravitational radiation that we can try to observe today. It is possible to calculate the quantum mechanical evolution of these fluctuations during inflation and the subsequent expansion of the universe until the present day. The present calculation of this evolution directly exposes the particle creation during accelerated expansion and while a fluctuation is larger than the Hubble length. Because all fluctuations regardless of their scale today began as the vacuum state in the early universe, the current quantum mechanical state of fluctuations is correlated on different scales and in different directions.

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1. Introduction

When one thinks of macroscopic manifestations of quantum mechanics, one usually thinks of phenomena such as superconductivity and lasers that have crept at least somewhat into our everyday lives. However, if there was an inflationary epoch in the early universe, quantum mechanical fluctuations and correlations determined the large-scale structure of the universe, and galaxies and superclusters are quantum mechanics writ upon the largest scales of our Universe.

During inflation, a slowly rolling scalar field (the inflaton) drives an exponential expansion of spacetime. This rapid expansion addresses several cosmological issues. It ensures that space is very close to flat and that topological defects that form before and early during inflation will be exceedingly rare today. Finally, the quantum mechanical evolution of the perturbations in a spatially uniform inflaton field (or other fields) provides the seeds for structure formation in the recent universe and for the fluctuations of the microwave background.

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There are several equivalent ways to calculate the evolution of fluctuations in scalar fields during the approximately de Sitter inflationary period. Possibly, the most straightforward way is to exploit the fact that the equation describing the evolution of the field operator is the same as that describing the classical evolution of a scalar field [1]; one can start with a reasonable approximation for the wavefunction of the de Sitter vacuum and evolve it forward through the de Sitter stage. At the onset of radiation domination, one can use the method of Bogolubov coefficients [2] to ensure continuity of the field and its time derivative.

Studies of how fluctuations in the early universe lose coherence have followed two main paths. The first looks at a quantum field coupled to the environment: for example, a thermal bath [3] or the backreaction of pair production of the quantum state of the universe [4]. The second path is to examine how the appearance of decoherence develops in the de Sitter space without any decohering interaction. Work to understand how initially quantum fluctuations begin to act classically as they are stretched outside the Hubble length moved forward dramatically with the work of Guth and Pi [5]. They found that long after a mode passes through the Hubble length, it follows a classical Gaussian probability distribution. Several people have addressed the quantum-to-classical transition in terms of squeezed states [6–8]. Albrecht *et al* [8] argued that although the the concept of squeezed states is useful in understanding the evolution of density fluctuations, it is entirely equivalent to more traditional methods and does not provide any new physical consequences.

This paper presents an alternative method that explicitly follows the particle creation. It is perhaps closest in spirit to the recent work of Mijić [9] that follows the evolution of the particle number as a mode is stretched beyond the Hubble length. Rather than treat the evolution of the field in the standard manner using a field operator that follows the de Sitter vacuum, we can use an operator that allows the wavefunction to expand as a superposition of multi-particle states as observed by a small comoving detector. The entire wavefunction is calculated as a function of time rather than the expectation value of a particular operator (typically the field operator). This method is computationally intensive, so it cannot be applied to modes that spend many Hubble times larger than the Hubble length. However, it offers new insights into how density fluctuations approach the classical limit.

In particular one can calculate the particle number using both techniques and they agree, but one can also construct any operator after the wavefunction has been calculated and determine its expectation value or the probabilities of the various outcomes of a particular measurement. Section 2 will derive the Hamiltonian of a scalar field in a homogeneous, isotropic universe and define the field operators for the de Sitter vacuum (section 2.1) and the vacuum of a radiation-dominated universe (section 2.2). The evolution of the wavefunction is described in section 2.3 and its initial conditions in section 2.4.

The results of the calculation (section 3) fall in two categories. First, we examine the robustness of the calculations and compare with the results from the standard treatments in section 3.1. We find that this treatment predicts the same amount of particle production as in the standard treatment (section 2.1 and e.g. [10]) and that the number of particles follows a thermal distribution. This highly excited Fock state behaves quasi-classically; however, the state of the field is still pure and furthermore the phases of the various multiparticle states are correlated (section 3.2). Section 5 speculates on the possibility of observing these correlations and future work.

2. The evolution of scalar-field modes

We will follow the evolution of perturbations of a uniform real scalar field in a homogeneous and isotropic spacetime with flat spatial sections. The metric is

$$ds^2 = a^2(\tau)(d\tau^2 - d\mathbf{x}^2), \tag{1}$$

where $a(\tau)$ is the scale factor. The action for the scalar field (ϕ) is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right], \tag{2}$$

where we have taken $\hbar = c = 1$ and $\sqrt{-g} = a^4(\tau)$. We neglect the backreaction of the perturbations on the metric. From the action, we would like to find the Hamiltonian and determine the quantum mechanical evolution of the fluctuations.

Now let us take $V(\phi) = \frac{1}{2}(m^2+\xi R)\phi^2+V_0$, where the scalar field may have a non-minimal coupling to gravity through the Ricci scalar R [11]. With these, we obtain

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left[\partial_\mu\phi\partial^\mu\phi - (m^2+\xi R)\phi^2\right] \tag{3}$$

$$= \frac{1}{2}\int d^4x a^2\left[\left(\frac{\partial\phi}{\partial\tau}\right)^2 - (\nabla\phi)^2 - a^2\left(m^2 + \frac{6\xi}{a^3}\frac{\partial^2 a}{\partial\tau^2}\right)\phi^2\right] \tag{4}$$

where we have dropped the constant term V_0. Let us make the substitution $u = a\phi$ to try to absorb the factor of $a^2(\tau)$ into the fields

$$S = \frac{1}{2}\int d^4x\left[\left(a\frac{\partial(u/a)}{\partial\tau}\right)^2 - (\nabla u)^2 - a^2\left(m^2 + \frac{6\xi}{a^3}\frac{\partial^2 a}{\partial\tau^2}\right)u^2\right] \tag{5}$$

$$= \frac{1}{2}\int d^4x\left[\left(\frac{\partial u}{\partial\tau} - \frac{\partial a}{\partial\tau}\frac{u}{a}\right)^2 - (\nabla u)^2 - a^2\left(m^2 + \frac{6\xi}{a^3}\frac{\partial^2 a}{\partial\tau^2}\right)u^2\right] \tag{6}$$

$$= \frac{1}{2}\int d^4x\left[\left(\frac{\partial u}{\partial\tau}\right)^2 + \frac{d}{d\tau}\left(-\frac{u^2}{a}\frac{\partial a}{\partial\tau}\right) + \frac{1}{a}\frac{\partial^2 a}{\partial\tau^2}u^2 - (\nabla u)^2 - a^2\left(m^2 + \frac{6\xi}{a^3}\frac{\partial^2 a}{\partial\tau^2}\right)u^2\right] \tag{7}$$

$$= \frac{1}{2}\int d^4x\left[\left(\frac{\partial u}{\partial\tau}\right)^2 - (\nabla u)^2 - \left(a^2m^2 + \frac{6\xi - 1}{a}\frac{\partial^2 a}{\partial\tau^2}\right)u^2\right], \tag{8}$$

where to get the final result we have dropped a term in the integrand equal to a total derivative with respect to the conformal time.

The field $u(\mathbf{x}, \tau)$ can be expressed in terms of Fourier modes:

$$u(\mathbf{x}, \tau) = \frac{1}{(2\pi)^{3/2}}\int d^3 k u_{\mathbf{k}}(\tau)\, e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{9}$$

where $u_{-\mathbf{k}}(\tau) = u^*_{\mathbf{k}}(\tau)$ to ensure that the field u is real. This yields

$$S = \frac{1}{2}\int d\tau\, d^3k\left[\left|\frac{\partial u_{\mathbf{k}}}{\partial\tau}\right|^2 - \left(k^2 + m_{\mathrm{eff}}^2\right)|u_{\mathbf{k}}|^2\right], \tag{10}$$

where $\mathbf{k}$ is the comoving momentum of a mode.

Now we have the action for a related scalar field u in a flat spacetime with a negative contribution to the square of the mass of the mode [1, 12].

The value of the mass depends on the evolution of the background spacetime (i.e. $a(\tau)$). If the dominant energy density in the universe is characterized by a pressure that is proportional to the energy density ($P = w\rho$), the scale factor evolves as

$$a(\tau) \propto \begin{cases} e^{(aH)\tau} & \text{if} \quad w = -\frac{1}{3} \\ \tau^{2/(3w+1)} & \text{otherwise} \end{cases}, \tag{11}$$

where H is the Hubble parameter, $(\partial a/\partial\tau)/a^2$. For $w = -1/3$, the product of the scale factor and the Hubble parameter is constant. The effective mass is

$$m_{\mathrm{eff}}^2 = -2\frac{Q}{\tau^2} \tag{12}$$

with

$$Q \equiv \frac{1}{(1+3w)^2}\left[(1-3w)(1-6\xi) - 2\frac{m^2}{H^2}\right] \tag{13}$$

if $w \neq -1/3$. The special case $w = -1/3$ yields

$$m_{\mathrm{eff}}^2 = (aH)^2\left(6\xi - 1 + \frac{m^2}{H^2}\right). \tag{14}$$

The Hamiltonian of the field is given by

$$H = \frac{1}{2}\int \mathrm{d}^3k\left[\left|\frac{\partial u_{\mathbf{k}}}{\partial\tau}\right|^2 + \left(k^2 + m_{\mathrm{eff}}^2\right)|u_{\mathbf{k}}|^2\right]. \tag{15}$$

To look at the quantum mechanics of this Hamiltonian one can take one of several routes, depending on the definition of the field operator $\hat{u}_{\mathbf{k}}$. In general, it is [1]

$$\hat{u}_{\mathbf{k}} = g(k,\tau)\hat{a}_{\mathbf{k}} + g^*(k,\tau)\hat{a}_{-\mathbf{k}}^\dagger, \tag{16}$$

where $\hat{a}_{\mathbf{k}}^\dagger$ and $\hat{a}_{\mathbf{k}}$ are creation and annihilation operators, respectively, that satisfy the following standard commutator relations

$$\left[\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{k}'}^\dagger\right] = \left[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}\right] = 0, \qquad \left[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger\right] = \delta^{(3)}(\mathbf{k} - \mathbf{k}'). \tag{17}$$

There is some flexibility in choosing the functions $g(k,\tau)$ or equivalently defining the states upon which the creation and annihilation operators act.

Specifically, one can choose $g(k,\tau)$ so that the Hamiltonian operator commutes with the particle number operator, $\hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{k}}$, and all of the time dependence is carried by the field operator or one can choose $g(k,\tau)$ so that the number of particles changes. The first option is tractable in a closed form if Q is constant or piecewise constant in τ, that is if $m \neq 0$ and $w = -1$ or $m = 0$ and w is constant or piecewise constant.

2.1. Vacuum picture

In the first case, the Hamiltonian operator is given by

$$\hat{H} = \frac{1}{2}\int \mathrm{d}^3k\, k\left(\hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{k}}^\dagger\right). \tag{18}$$

The Hamiltonian commutes with the particle number operator, so the universe remains in the vacuum state, $|0\rangle$. If the Hamiltonian takes this form, the functions $g(k,\tau)$ must satisfy the following differential equation:

$$\frac{\partial^2 g(k,\tau)}{\partial\tau^2} + \left(k^2 - 2\frac{Q}{\tau^2}\right)g(k,\tau) = 0. \tag{19}$$

If Q is constant, we have

$$g(k,\tau) = \sqrt{\tau}[AJ_\nu(k\tau) + BY_\nu(k\tau)], \tag{20}$$

where $\nu = \sqrt{1+8Q}/2$. Two solutions of particular importance are for $Q = 1$ for $w = 0, -1$ and $\xi = m = 0$ and $Q = 0$ for $w = 1/3$ and $\xi = m = 0$. We have

$$g(k,\tau) = \frac{1}{\sqrt{2k}}\,\mathrm{e}^{-\mathrm{i}k\tau} \tag{21}$$

for $Q = 0$ and

$$g(k,\tau) = \frac{1}{\sqrt{2k}}\,(k\tau - i)\,\frac{\mathrm{e}^{-\mathrm{i}k\tau}}{k\tau} \tag{22}$$

for $Q = 1$ [1]. For $w = -1$ (de Sitter space), one can use the general solution, equation (20), when $m \neq 0$ and $\xi \neq 0$, and during radiation domination $\left(w = \frac{1}{3}\right)$, one can find a general solution in terms of Whittaker functions even when $m \neq 0$ and $\xi \neq 0$. In this paper the focus will be massless scalar fields, i.e. perturbations to the inflaton field and gravitational waves that can be modelled as two independent massless scalar fields [2]; therefore, the solutions given by equations (21) and (22) will suffice for the radiation-dominated and de Sitter phases, respectively.

In general if Q is only piecewise constant, one must insist that the function $g(k,\tau)$ and its first derivative are continuous through the transition, because equation (19) is second order. This determines the values of A and B in equation (20) after the transition (i.e. the Bogolubov transformation).

In particular if we look at the transition from vacuum domination to radiation domination, we find that equation (22) applies during vacuum domination and using the general solution, equation (20), we find that

$$g(k,\tau) = \frac{1}{\sqrt{2k}}\frac{x_{\mathrm{max}}^2}{2}\left[\left(2x_{\mathrm{max}}^{-2} - 2\mathrm{i}x_{\mathrm{max}}^{-1} - 1\right)\mathrm{e}^{-\mathrm{i}k\tau} + \mathrm{e}^{-2\mathrm{i}/x_{\mathrm{max}}}\,\mathrm{e}^{\mathrm{i}k\tau}\right] \tag{23}$$

during the subsequent radiation-dominated epoch where $x_{\mathrm{max}} = 1/(k\tau_{\mathrm{RH}}) = a_{\mathrm{RH}}H/k$, the ratio of the Hubble parameter to the physical wavenumber at the end of inflation.

From equation (23), we can read off the number of particles that are created in a particular mode from an initial vacuum state; it is simply given by the square of the coefficient of the negative frequency term (see, for example, equation 1.26 of [10]),

$$\langle N\rangle = \frac{1}{4}\left(\frac{a_{\mathrm{RH}}H}{k}\right)^4. \tag{24}$$

2.2. *Particle picture*

In this second technique, one chooses a particular function for $g(k,\tau)$ and sticks with it for the entire calculation. Specifically, we choose equation (21). This has two advantages. First, this choice of $g(k,\tau)$ is appropriate for radiation domination, i.e. the state of the universe after inflation, so the number of particles in a particular state remains constant after inflation. Second, this $g(k,\tau)$ is appropriate for modes whose physical wavelengths are much smaller than the Hubble length, i.e. the situation in the distant past and the distant future (today). The particle number operator, $\hat{a}_{\mathbf{k}}^{\dagger}\hat{a}_{\mathbf{k}}$, does not commute with the Hamiltonian during the de Sitter epoch, and its expectation value gives the number of particles that we would expect to observe with a particle detector looking for excitations which are small compared to the present-day scale of the universe. Although there is considerable freedom in choosing the functions $g(k,\tau)$ due to the vacuum ambiguity that appears in the quantum field theory in curved spacetime, this particular choice of $g(k,\tau)$ gives the canonical quantization of a scalar field in a flat spacetime, in particular the locally flat, inertial frame of a comoving detector, imparting some conceptual advantages of this picture over that standard picture outlined in section 2.1.

Because the Lagrangian for a scalar field in a Robertson–Walker spacetime evolves in the same way as a scalar field in a flat spacetime but with a time-dependent mass, it is quite natural to express the Hamiltonian for the system as the sum of the flat space Hamiltonian [13] and time-dependent portion:

$$\hat{H} = \frac{1}{2}\int \mathrm{d}^3k\left[\left(k - \frac{Q}{\tau^2 k}\right)\left(\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}\right) - \frac{Q}{\tau^2 k}\left(\hat{a}_{-\mathbf{k}}\hat{a}_{\mathbf{k}}\,\mathrm{e}^{-2\mathrm{i}k\tau} + \hat{a}^\dagger_{-\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}\,\mathrm{e}^{2\mathrm{i}k\tau}\right)\right]. \tag{25}$$

$\hat{a}_{\mathbf{k}}$ annihilates the vacuum of a radiation-dominated universe, $\hat{a}_{\mathbf{k}}\,|0\rangle = 0$.

The advantage of this technique is that it connects an initial state (the vacuum) to the final state in the basis in which we observe it. Because the final state is effectively the large-scale structure of the universe, following the quantum mechanical evolution of the scalar field becomes exponentially cumbersome for modes much larger than the Hubble scale. Nevertheless, even the short-term evolution of modes as they expand beyond the Hubble scale provides new insights.

2.3. Radiation-dominated spacetime Fock space

The Hamiltonian consists of a particle-conserving portion (which is the sum of the radiation-dominated result and a time-dependent correction) and a component which creates and destroys particles. The entire Hamiltonian commutes with the total momentum, so it is natural to examine how the Hamiltonian acts on states with zero total momentum. Let us take the following sum of states:

$$|\psi\rangle = \sum_{n=0}^{\infty} B_n(\tau)\frac{\left(\hat{a}^\dagger_{\mathbf{k}'}\right)^n\left(\hat{a}^\dagger_{-\mathbf{k}'}\right)^n}{n!\left[\hat{a}_{\mathbf{k}'}, \hat{a}^\dagger_{\mathbf{k}'}\right]^n}|0\rangle \tag{26}$$

$$= \sum_{n=0}^{\infty} B_n(\tau)|n, -\mathbf{k}'; n, \mathbf{k}'\rangle, \tag{27}$$

where we have assumed that the vacuum is normalized.

$|B_n(\tau)|^2$ is the probability of a comoving observer in the distant future detecting n particles with comoving momentum $\mathbf{k}'$ and n particles with momentum $-\mathbf{k}'$ if inflation ended at a conformal time τ. This yields

$$\hat{H}|\psi\rangle = \sum_{n=0}^{\infty} B_n(\tau)\left[\left(k - \frac{1}{\tau^2 k}\right)(2n+Z)|n, -\mathbf{k}; n, \mathbf{k}\rangle - \frac{1}{\tau^2 k}\left(n|n-1, -\mathbf{k}; n-1, \mathbf{k}\rangle\,\mathrm{e}^{-2\mathrm{i}k\tau}\right.\right.$$
$$\left.\left. + (n+1)|n+1, -\mathbf{k}; n+1, \mathbf{k}\rangle\,\mathrm{e}^{2\mathrm{i}k\tau}\right),\right. \tag{28}$$

where $Z = \left[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}}\right] = \delta^{(3)}(\mathbf{k} - \mathbf{k})$ is an infinite constant related to the vacuum energy of the system. Its value is independent of the number of particles in the states.

We have replaced $\mathbf{k}'$ with $\mathbf{k}$ after performing the integral over all momenta in equation (26). The integral with our choice of basis functions is manifestly even in $\mathbf{k}$, so the factor of one-half vanishes. Furthermore, the Hamiltonian yields a solution separable in momentum space, so we can solve for the evolution of each value of comoving momentum separately. As shown in the subsequent paragraphs, by solving this for a single value of the comoving momentum in the de Sitter space one obtains solutions useful over a range of momenta.

The time evolution of the coefficients $B_n(\tau)$ is given by the value of

$$\langle n, -\mathbf{k}; n, \mathbf{k}|\hat{H}|\psi\rangle = \mathrm{i}\frac{\mathrm{d}B_n(\tau)}{\mathrm{d}\tau} \tag{29}$$

$$= \left[\left(k - \frac{Q}{\tau^2 k}\right)(2n + Z)B_n(\tau) - \frac{Q}{\tau^2 k}(B_{n-1}(\tau)n\,\mathrm{e}^{2\mathrm{i}k\tau} + B_{n+1}(\tau)(n+1)\,\mathrm{e}^{-2\mathrm{i}k\tau})\right]. \tag{30}$$

Since $\hat{H}$ evolves a state forward in time, we obtain

$$\mathrm{i}\frac{\mathrm{d}A_n(x)}{\mathrm{d}x} = -Q[mA_{n-1}(x)\,\mathrm{e}^{-2\mathrm{i}\gamma(x)/x} + (n+1)A_{n+1}(x)\,\mathrm{e}^{2\mathrm{i}\gamma(x)/x}], \tag{31}$$

where we changed variables to $x = -1/(k\tau)$ and absorbed an infinite phase into the definition of A_n:

$$A_n(x) = \mathrm{e}^{-\mathrm{i}(2n+Z)(\gamma(x)-1)/x}B_n(x) \tag{32}$$

$$\gamma(\tau) = 2 + Qx^2. \tag{33}$$

The variable x equals aH/k during the de Sitter phase and $-aH/k$ during radiation domination. The value of x increases with time during both these phases. The physical momentum p is given by $p = k/a$, so $p/H = 1/|x|$. The value of Q during the vacuum-energy-dominated phase ($w = -1$) is unity. During radiation domination ($w = 1/3$), in the massless case Q vanishes, so the coefficients A_n are constant during the radiation-dominated epoch (see equation (31)).

It is important to keep in mind that the functional relationship between the effective mass and the conformal time dictated both the change of variables and the (infinite) phase shift of the wavefunctions. In a situation where Q is not constant with respect to τ, a different set of substitutions is necessary. In general, we must integrate up the time dependence of the effective mass (equation (32)) to renormalize the infinite zero-point energy of the field. This could be done numerically of course.

Because $A_n(\tau)$ differs from B_n only by a phase, $|A_n(\tau)|^2$ is still the probability of a comoving observer detecting n particles with comoving momentum $\mathbf{k}$ and n particles with comoving momentum $-\mathbf{k}$ during radiation domination or the distant future if inflation ends at conformal time τ. Furthermore, because the differential equation only depends on the product of $k\tau$, the solution to the equation gives the wavefunction in the Fock space of a radiation-dominated spacetime for a range of comoving momenta.

2.4. Initial conditions

Initially, well before the mode crosses outside of the Hubble length, $|A_0(\tau \to -\infty)|^2 = 1$ and there are no particles. During inflation, the mode expands relative to the Hubble length. At the end of inflation, $\tau = -\tau_{\mathrm{RH}}$, the universe reheats and the energy in the inflaton field decays into relativistic particles. After reheating, we can no longer follow the quantum fluctuations of the inflaton field itself; however, any correlations in the inflaton field should also be present in the resulting perturbation to the density of relativistic matter. The quantum fluctuations of a fluid follow equations similar to those of the scalar field [12]; specifically the value of Q is the same for scalar fields and fluids. We can also follow the evolution of the perturbations in the second scalar field which does not couple to the inflaton as they cross back within the Hubble length. The perturbations induced by inflation on the metric itself can be decomposed into two minimally coupled scalar fields [2], so the primordial spectrum of gravitational radiation falls into this category and furthermore these fluctuations remain in the linear regime until the present epoch.

For simplicity, the vacuum and radiation-dominated regimes are pasted together so that after the end of inflation $\tau = \tau_{\rm RH}$, i.e. we assume that reheating is instantaneous and that the universe does not expand during this time. The fact that inflation ends suddenly results in production of particles at all momenta. Specifically, the total energy of the particles produced diverges in the high-frequency limit; however, the sudden approximation is only valid for modes whose periods are much greater than the timescale for the transition. Higher frequency modes will evolve adiabatically, so their final state will be the vacuum state. We also neglect any nonlinear couplings that may develop during reheating. This is probably reasonable for tensor fluctuations, but nonlinearities may play an important role for density fluctuations and this is an avenue for further work.

The conformal time, τ, continues to increase until the present day. The number of states occupied as well as the mean number of particles per mode increases dramatically with the time that a particular mode spends outside of the Hubble length during an inflationary period. On the other hand, during radiation domination, the wavefunctions evolve only trivially; the coefficients A_n are constant.

3. Results

As discussed in the previous paragraph, the natural initial condition for the simulation when a mode is much smaller than the Hubble length is $|\psi\rangle = |0, \mathbf{k}; 0, -\mathbf{k}\rangle$. We simulated the evolution of a massless real scalar field using this initial condition and equation (31). To assess the accuracy of the simulations, the initial states $|1, \mathbf{k}; 1, -\mathbf{k}\rangle$ and $|2, \mathbf{k}; 2, -\mathbf{k}\rangle$ are also evolved forward. The inner product between these states and the initial vacuum state vanishes initially.

3.1. Number of particles

As figure 1 depicts, each initial state results in thousands of particles being produced over a wide range of comoving momenta. The rate of particle production is dramatically larger than that found by Mijić [9]. Also these calculations do not exhibit the discontinuity that Mijić found at $H/p = 1/\sqrt{2}$ when the mass of a mode becomes imaginary. Although the pair production is prodigious and the final states have little overlap with the initial states, the accumulated numerical error estimated by the scalar products and the conservation of the norm of the states is modest (figure 2). The accumulated error increases in a series of periodic steps rather than continuously. Each of these steps corresponds approximately to a conformal time where $2\gamma(x) = n\pi$, where n is a negative integer (see equation (33)).

For all three sets of initial conditions, the number of particles increases dramatically for the modes that have exited through the Hubble length. However, in no case does the particle number increase exponentially with the comoving momentum; this would be indicated by a straight line in figure 1.

The simulations show that the occupation of the states is well described by a thermal distribution (figure 3), Such a distribution may be characterized by a single parameter, the expectation value of n. The total number of particles is, of course, $N = 2n$. Figure 4 depicts the expectation value of $2n$ as a function of the ratio of the physical momentum to the Hubble

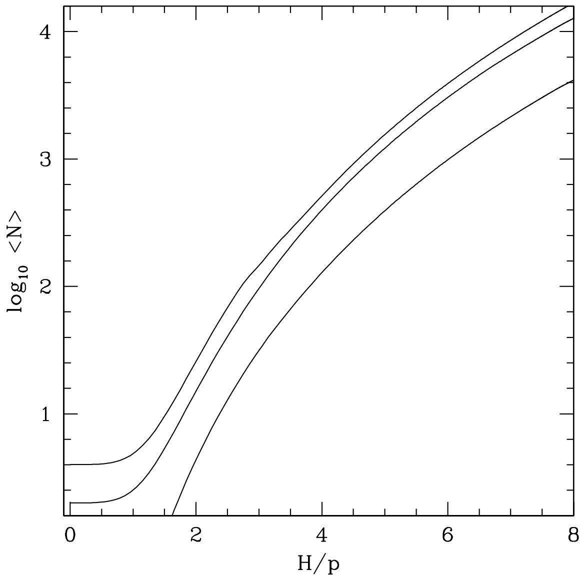

Figure 1. Evolution of the expectation value of the particle number before and soon after the Hubble length exit. The upper curve is for a four-particle initiastill l state, the middle curve is a two-particle initial state and the lower curve is a vacuum initial state.

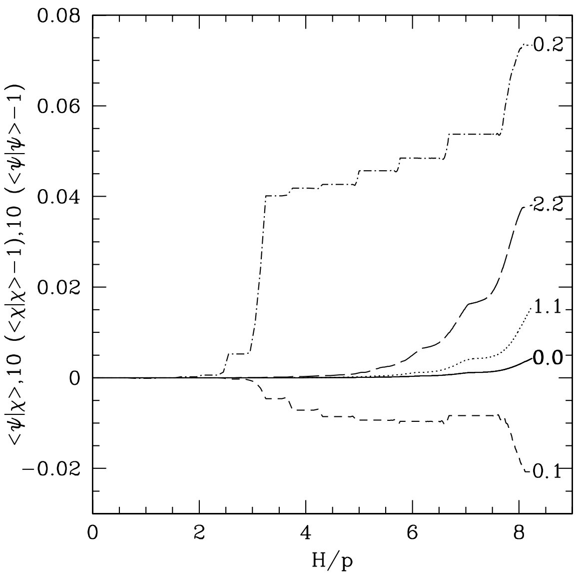

Figure 2. Evolution of the scalar product between the various initial states. For clarity, the value of $10\left(\sum A_n^* A_n - 1\right)$ is depicted as well as the cross products. If the simulation were free of numerical errors, these values would all vanish.

parameter at the end of inflation. We have from equation (24) the number of particles in a particular mode:

$$\langle N \rangle \approx \frac{1}{4}\left(\frac{p}{H}\right)^{-4}. \tag{34}$$

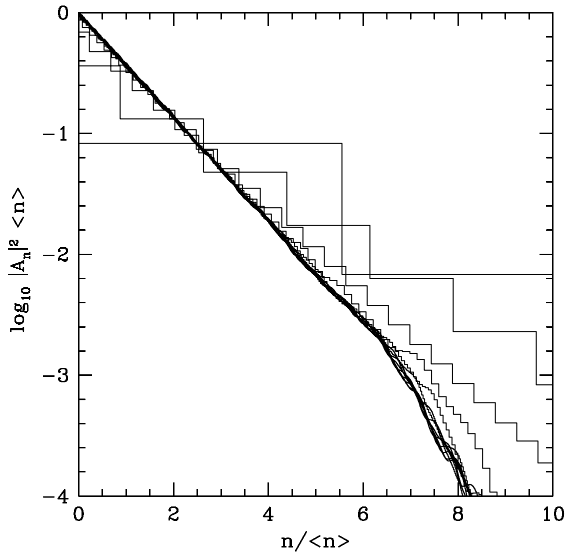

Figure 3. The curves trace the final values of $|A_n|^2$ as a function of $n/\langle n\rangle$. The curves are normalized by the expectation value of n. Starting at the lower right corner of the panel and moving left, the curves trace modes with the following maximum values of $x \equiv H/p$, 1.0, 1.5, 2.0 . . . 8.0. The distributions for $x > 3$ are nearly identical.

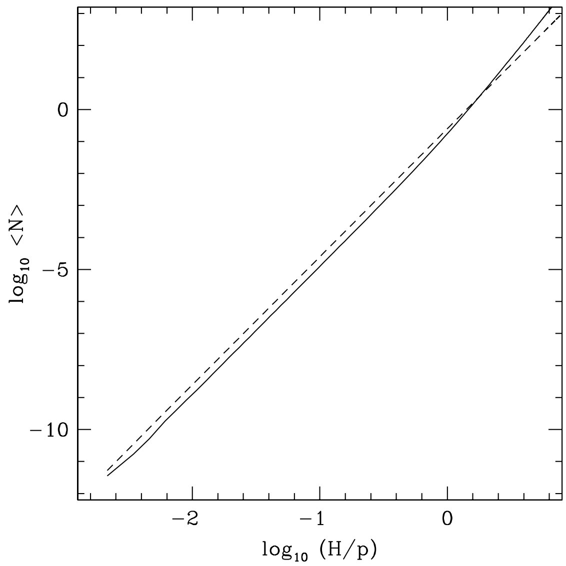

Figure 4. The solid line traces the calculated value of $\langle N\rangle = \langle 2n\rangle$ as a function. The ratio of the physical momentum to H is the Hubble parameter at the end of inflaton in the particle picture. The figure is similar to figure 1, but focuses on high frequency modes. The dashed line shows the value given by equation (24) in the vacuum picture.

Because the field theory for the field u is defined for a comoving volume, the physical energy density of particles produced within a range of physical momenta dp is

$$\frac{\mathrm{d}E}{\mathrm{d}V\,\mathrm{d}p} \approx \frac{1}{8\pi^2}\frac{H^4}{p}, \tag{35}$$

so the total energy density of the modes diverges as $\ln p_{\max}/p_{\min}$ where $p_{\max} \approx 1/\Delta t$ is some momentum cutoff, set by the duration of reheating [10]. If the product of the physical momentum and the duration of reheating is much greater than 1, the value of Q changes sufficiently slowly so that the initial vacuum oscillation will evolve adiabatically to the final vacuum of the radiation-dominated epoch. The infrared momentum cutoff is given approximately by the Hubble constant during reheating, $p_{\min} \approx H$ [10].

3.2. Primordial correlations

At some point, the expansion of the universe becomes radiation dominated. If we assume that this transition is abrupt, we can calculate the subsequent evolution of a mode during the period of radiation domination as it shrinks relative to the Hubble length. The coefficients A_n remain constant during this time because $Q = 0$ (see equation (31)); in particular, the number of particles remains constant. When matter begins to dominate the energy density, the coefficients A_n again change with time as long as the mode lies outside the Hubble length because now $Q \neq 0$. The focus here will be modes that only spend at most a few Hubble times outside the Hubble length, so they will be well within the Hubble length at the onset of matter domination, and the values of A_n at reheating will characterize the fluctuations today.

Even for modes that only spend a short time outside the Hubble length, the final states long after reheating contain a large number of particles (figure 1). However, the partition function for a given mode remains thermal. Figure 3 shows that a simple rescaling of the distributions by the mean number of particles maps the final distributions into each other.

Although the partition function for the number of particles is thermal for the final state of the fluctuations, the final wavefunctions for different values of the comoving momentum are correlated.

The modes approach the classical regime when $x_{\max}$ gets larger, as is shown by figure 5. The figure depicts the scalar product between the probability amplitudes of the final wavefunctions of modes that have spent nearly the same amount of time outside the Hubble length or have slightly different values of k. We find that for modes that only spend a short time outside the Hubble length, the final state is quite similar over a wide range of momentum. The range over which the modes correlate decreases as $x_{\max}^{-3}$ for $x_{\max} \gg 1$. As $x_{\max}$ increases, the correlation between the modes diminishes. An important fact to keep in mind is that although the obvious correlations between the wavefunctions vanish as $x_{\max}$ gets larger, the wavefunctions are still correlated albeit more subtly; the phases of the probability amplitudes are not strictly random but pseudo-random, that is to say calculable.

Unfortunately, these correlated quantum phases are not directly observable. The wavefunction for each value of the comoving momentum resides in a separate Hilbert space, so our observations of the fluctuations on these small scales are drawn from the same underlying distribution but are otherwise independent. Only the most gross properties of the distribution are correlated such as the probability to find a certain number of particles in a particular state. On the other hand, the correlated phases have a great pedagogical value in that they verify that the production of fluctuations during inflation does not require the destruction of information; it is strictly unitary.

4. Sensitivity to the details of reheating

Although the simulations here assumed that reheating quickly converts the potential energy of the scalar field to the kinetic energy of relativistic particles, simulations of a transition to a matter-dominated regime yielded very similar results. The final distributions are still

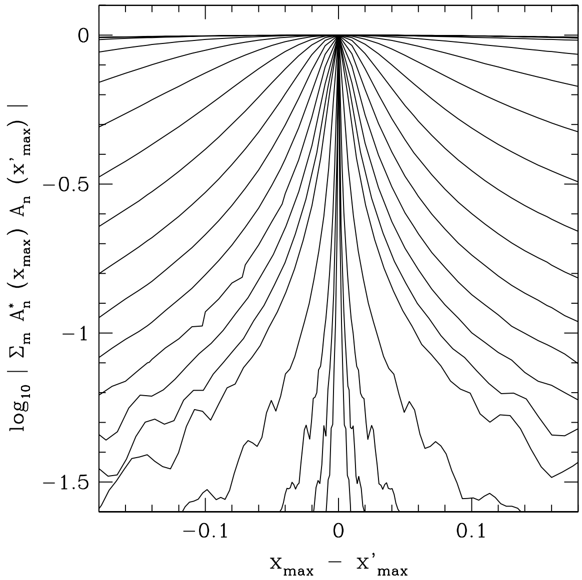

Figure 5. The final value of $\sum_n A_n^* A_n'$ for different values of $x_{\max} = (H/p)_{\max}$. Starting with the uppermost curve on the left $x_{\max} = 1.6, 1.8, 2.0 \ldots 4.4, 5, 6, 7, 8$.

thermal as in figure 3 and the correlations between the quantum states of modes with different values of **k** are similar to figure 5 except that in the case of a matter-dominated reheating, the correlations are typically weaker for given values of **k** and **k**′.

Of course, this is not a general demonstration that these results are robust with respect to the details of reheating but it does indicate that quantum correlations may persist in the present day.

5. Discussion

The foregoing results show that fluctuations on scales of the present-day universe that passed through the Hubble length near the end of the inflationary epoch exhibit quantum mechanical correlations that may belie their birth from the vacuum. Today, the comoving scale of the Hubble length at the end of inflation is

$$\frac{1}{k_{\rm RH}} = 10^5 \left(\frac{M}{10^{14}\,{\rm GeV}}\right)^{2/3} \left(\frac{T_{\rm RH}}{10^{10}\,{\rm GeV}}\right)^{1/3} {\rm cm}, \qquad (36)$$

where M^4 is the vacuum energy associated with the inflaton field and $T_{\rm RH}$ is the temperature of the universe at the end of reheating. The analysis has assumed that reheating is quick and efficient [14, 15] so $T_{\rm RH} \sim M$, yielding

$$\frac{1}{k_{\rm RH}} = 2 \times 10^6 \left(\frac{M}{10^{14}\,{\rm GeV}}\right)^1 {\rm cm}. \qquad (37)$$

The value of $x_{\rm RH}$ for the comoving scale $1/k_{\rm RH}$ is simply 1 and $k = k_{\rm RH}/x_{\max}$. Consequently although the correlations are present on all scales, they are most obvious on the comoving scale of the Hubble length at the end of inflation (i.e. really small scales). On these small scales the density fluctuations are well into the nonlinear regime today but tensor fluctuations, gravitational waves (GW), would still be a loyal tracer of these correlations.

The expression given in equation (37) is very uncertain. A simple way to quantify the chance that we might observe these correlations directly is to calculate the number of e-foldings between the time when our present Hubble scale equalled the Hubble scale during inflation and the typical scale of future GW observatories probing inflation (e.g. the Big-Bang Observatory [16]):

$$\Delta N = \ln \frac{0.1\mathrm{Hz}}{H} \approx 38. \tag{38}$$

Typically today's Hubble scale is assumed to pass out through the Hubble length during inflation about 50− 60 e-foldings [1] before the end, so the millihertz scale would pass through the Hubble length 12− 22 e-foldings before the end. However, the former number is highly uncertain. For example, if inflation occurs at a higher energy scale or if there is an epoch of late 'thermal inflaton' [17–19], the number of e-foldings for today's Hubble scale could be as low as 25 [1].

Even if we eventually observe the fluctuations on the comoving scales corresponding to the Hubble length at the end of reheating, one could argue that the wavefunction only determines the probability of measuring a particular amplitude and phase for the primordial gravitational waves at a particular momentum, and one would need to measure these fluctuations in several different realizations (universes) to unearth the correlations.

The value of these correlations lies in that they verify that the production of fluctuations during inflation does not require the destruction of information; it is strictly unitary. Furthermore, one can use the wavefunction to calculate the density matrix for observations restricted to a portion of the spacetime (e.g. within the horizon [20]) and estimate the entanglement entropy of the observable fluctuations. The calculations can also be easily extended to include nonlinearities in the field either through gravity or through self-interactions. Again these interactions generate entanglement entropy when fluctuations on a single scale are observed, giving the appearance of entropy production while the scale field remains in a pure state. Both of these calculations will be addressed in a future paper.

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